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HYDROGASDYNAMICS IN TECHNOLOGICAL PROCESSES

SPIRAL VORTICES IN A CREEPING FLOW

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Using particular solutions of the Stokes equations as an example, it is shown that there exists a spiral vortex in a creeping flow. The solutions are obtained with the aid of a local theory making it possible to lower the dimensionality of the problem to unity as a result of employing a spiral coordinate and coordinate expansion in the vicinity of the singularity, with the axial coordinate "being frozen," i.e., entering into the equation as a parameter. The singularities of the type of focus and of limiting cycle are considered.

Keywords: *focus, limiting cycle, Stokes equation, monopole, columnar vortex, conical vortex.*

Introduction. The word *spiral*, meaning a coil, is pronounced similarly in Latin (spira) and Greek (σπερα). Curves with at least one coil can be called spirals, since a strict mathematical definition of the spiral, even as a plane curve, merely does not exist [1].

Statement of the Problem. We will introduce the polar coordinates r and θ. A coil of a plane spiral $r = f(\theta, t)$ is the name given to any of its segment extending from the point $r_1 = f(\theta_1, t)$ to the point $r_2 = f(\theta_1 + 2\pi, t)$. We will consider spirals with an infinite number of coils — they have found wide application in physics. We will define the spiral as a continuous line without self-intersections that, while coiling around a continuous closed line (limiting cycle) or a point, performs an infinite number of coils. If in mathematical models the number of coils of a spiral is infinite [2], under real conditions spirals have a finite number of coils.

A spiral whose shape has the form $\theta = kr^{-m}$, where k and $m > 0$ are constants, is called *algebraic*. When $\theta \to \pm\infty$, $r \to 0$. This representation can be considered the first term of expansion of the function $\theta(r)$ in powers of r. The remaining terms of the series, which are small near the spiral center, play a role at a distance from it. For the right spiral $k > 0$, $0 \le \theta < \infty$, for the left one $k < 0$, $-\infty < \theta \le 0$.

The equation of the algebraic spirals that are wound on the limiting cycle $r = r_*(\theta)$ is representable in an implicit form: $\theta = k(r - r_*)^{-m}$. The spiral trajectories may encircle the limiting cycle on its one side or on both sides: external and internal. In the former case the spiral trajectories border on closed lines located on the other side of the limiting cycle.

The equation of the *logarithmic* spiral in polar coordinates is $\theta = k \ln r$. The range of change of the polar angle θ is $-\infty > \theta > \infty$. If $k < 0$, the spiral is a right one; if $k > 0$, it is a left one.

When a determining parameter changes from 0 to ∞, there are always three possibilities of asymptotic classification. An example is a liquid flow determined by the Re number: Re << 1, a creeping flow; Re = $O(1)$, a viscous liquid flow; Re >> 1, a boundary layer or an ideal liquid flow.

The asymptotic classification of spiral cores is carried out by the limiting value of the steepness $\sigma = \lim_{r\to 0} r(\partial f/\partial\theta)^{-1}$. Three possibilities arise:

1) $\sigma = O(1)$, a logarithmic spiral; in an asymptotic sense, it is an intermediate type of the spiral core, the spiral length from the center to the point $r = r_1$ is finite: $s = \int_0^{r_1} (1 + \sigma^2)^{1/2} dr = O(r_1)$;

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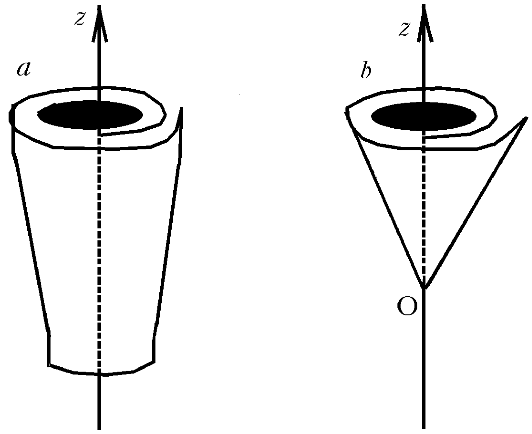

Fig. 1. Topological classification of three-dimensional vortices: a) columnar vortex; b) conical vortex.

2) $|\sigma| = \infty$, a steep spiral, rapidly converging to the center, the spiral length is also finite: $s = O(r_1)$, with the arc length equal to $s = \int_0^{r_1} \sigma dr$ (algebraic spirals belong to this type);

3) $\sigma = 0$, a sloping spiral, a multiturn one, slowly approaching the center (an example of a sloping spiral: $\theta = k \ln |\ln r|$).

As an example of a spiral surface we will select the surface that as its axis has the line the normal section to which cuts a spiral with a focus located on this axis. Topologically, two types of spiral surfaces are possible: columnar ones, when the vortex axis extends infinitely to both sides, having neither end nor beginning, and conical ones, when the vortex axis has its beginning O. Figure 1a shows the limiting cycle $r = r_*(\theta, z)$ in a columnar spiral vortex, and Fig. 1b demonstrates the limiting cycle $\vartheta = \vartheta_*(\varphi, R)$ in a conical one.

The equation of a columnar spiral surface has the form $\theta = \theta_0(\xi, z)$, where $\xi = r - r_*$; $\theta \to \infty$, if $\xi \to 0$. In the case of focal singularity $r_* = 0$. The surface is called cylindrical if $\partial\theta_0/\partial z = 0$. Then the limiting cycle is a round cylinder $r_* = \text{const}$.

The equation of the conical spiral surface has the form $\varphi = \varphi_0(\zeta, \vartheta)$, where $\zeta = R - R_*$; $\varphi \to \infty$, if $\vartheta \to 0$. In the case of focal singularity $R_* = 0$. The surface is called conical if $\partial\varphi_0/\partial R = 0$. Then the limiting cycle can be represented by a round cone $\vartheta_* = \text{const}$.

We will identify a spatial spiral line with the trajectory of the point moving around and along a certain axis approaching it infinitely. The deviation of the spatial curve from the plane is characterized by torsion. The spatial spiral with zero torsion is a plane spiral.

Local Theory. In a turbulent flow one observes organized and unorganized vortex structures. Examples of organized structures are coherent vortices, and an example of unorganized structure is an isotropic turbulence [3].

The vortex is the rotation. It means that in an organized vortex structure there must be an axis of rotation — the line (a point cannot be a center of rotation since neither the vortex monopole nor the Dirac magnetic monopole has been revealed in experiment [2]). The liquid moves around this axis along spiral trajectories, because circular rotation is only a particular (limiting) case of spiral rotation.

Since the structure is organized, in a small scale — in the vicinity of the rotation axis — the flow can be considered laminar. The spiral (nonaxisymmetric) structure of flow in this vicinity is being studied within the framework of the local theory. In contrast to the general theory that considers the flow as a whole, i.e., in the entire region of its existence, the local theory studies the flow in the vicinity of singularities — varieties of smaller dimensions (points, lines, surfaces). With the aid of the local theory one can find singularity in numerical calculation of a flow, as well as to establish the very fact of the existence of a spiral flow (in the small).

The local theory is always asymptotic, since it employs coordinate or asymptotic series. The coordinate series in a certain argument dependent on the distance to the singularity makes it possible to reduce the dimensionality of the problem by unity.

In the problem on the spiral vortex the axis of the latter or some surface (limiting cycle) is the singularity. To analyze the flow in the vicinity of the singularity, i.e., to apply the local theory, one should transform from coordinates taking small or large values to coordinates taking a finite value. This is done with the aid of spiral coordinates. In the case of a columnar vortex, such a coordinate is $\eta = \theta - \theta_0$ $(0 \le \eta \le 2\pi)$, and in the case of a conical vortex this is the coordinate $\mu = \varphi - \varphi_0$ $(0 \le \mu \le 2\pi)$.

Within the framework of the local theory it is natural to assume that as the singularity is approached, the flow, ignoring the influence of the far field, acquires a universal form. The stationary flow becomes conical, and the nonstationary one self-similar [4]. This assumption also makes it possible to reduce the dimensionality of the problem.

In a large scale (Re $\to \infty$), i.e., in the approximation of an inviscid liquid, the existence and structure of spiral vortices was established in [5]. In the case of the other limiting transition (Re $\to 0$), a spiral vortex was found by Hamel; however, his solution appeared nonphysical. Therefore the problem on the existence of spiral (i.e., depending on the spiral coordinate) vortices in a viscous fluid remains open.

The equations of a creeping flow have the form

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla p = \Delta \mathbf{u}\,, \quad \nabla \mathbf{u} = 0\,, \tag{1}$$

where $\mathbf{u}$ is the velocity vector equal to unity.

Columnar Vortex. We will consider the solution in the vicinity of the focus and limiting cycle.

Nonstationary focus. We will restrict ourselves to the self-similar stage of the formation of focal singularity in a plane flow. We introduce the self-similar variables $\xi = tr^{-2}$ and $0 \le \eta = \theta - \theta_0(z, \xi) \le 2\pi$.

We expand the solution of Eq. (1) in a cylindrical coordinate system into a series in powers of r:

$$u_r(r, \theta, z, t) = r^{\alpha} u(z, \eta, \xi) + o(r^{\alpha})\,, \quad u_\theta(r, \theta, z, t) = r^{\alpha} v(z, \eta, \xi) + o(r^{\alpha})\,,$$
$$p(r, \theta, z, t) = r^{\alpha-1} \chi(z, \eta, \xi) + o(r^{\alpha-1})\,. \tag{2}$$

The flow will be three-dimensional if the component of the longitudinal velocity u_z exerts its influence on mass transfer. Then, according to the continuity equation, we obtain

$$u_z(r, \theta, z, t) = r^{\alpha-1} w(z, \eta, \xi) + o(r^{\alpha-1})\,. \tag{3}$$

The solutions of Eqs. (2) and (3) are inhomogeneously self-similar, since the coordinates r and z have been transformed differently.

Consider the logarithmic spiral $\theta_0 = -\frac{1}{2}\sigma(z) \ln \xi$; we restrict the discussion to a brief commentary on other possibilities. For the logarithmic spiral the transition from the polar coordinates r, θ to the spiral coordinates r, η is performed by the formulas

$$\frac{\partial}{\partial \theta} = \frac{\partial}{\partial \eta}\,, \quad \left(\frac{\partial}{\partial r}\right)_\theta = \left(\frac{\partial}{\partial r}\right)_\eta + \frac{1}{2}\frac{\sigma}{r}\frac{\partial}{\partial \eta}\,.$$

In the case of a sloping spiral we have $\left(\frac{\partial}{\partial r}\right)_\theta \approx -\frac{\partial \theta_0(z, \xi)}{\partial r}\frac{\partial}{\partial \eta}$. This relationship results from the limiting transition $|\sigma| \to \infty$. In the case of a steep spiral ($|\sigma| << 1$) equations follow from Eq. (1) as a result of the other limiting transition $|\sigma| \to 0$. Consequently, the case of the logarithmic spiral is more general as compared to these limiting transitions.

Due to the fact that in the selected coordinate system the range of change of the angle θ has become infinite $(0 \le |\theta| \le \infty)$, the solution should be 2π-periodic in θ, i.e., one-sheeted. Such requirement is adequate to the condition of analyticity of solution at the boundaries of the range of change of the spiral coordinate η: the sought functions and all of their derivatives on the lines $\eta = 0$ and $\eta = 2\pi$ must coincide. For Eq. (1) it will suffice to assign six conditions of this type, for example:

$$u(z, 0, \xi) = u(z, 2\pi, \xi), \quad v(z, 0, \xi) = v(z, 2\pi, \xi), \quad w(z, 0, \xi) = w(z, 2\pi, \xi), \tag{4}$$
$$\gamma(z, 0, \xi) = \chi(z, 2\pi, \xi), \quad \frac{\partial u(z, 0, \xi)}{\partial \eta} = \frac{\partial u(z, 2\pi, \xi)}{\partial \eta}, \quad \frac{\partial w(z, 0, \xi)}{\partial \eta} = \frac{\partial w(z, 2\pi, \xi)}{\partial \eta}.$$

Having substituted (2) and (3) into (1), we obtain

$$\frac{\partial v}{\partial \eta} + \frac{\partial w}{\partial z} - 2\xi \frac{\partial u}{\partial \xi} - \sigma \frac{\partial u}{\partial \eta} + (1 + \alpha)\, u = 0,$$

$$\frac{\partial u}{\partial \xi} + \frac{1}{2}\frac{\sigma}{\xi}\frac{\partial u}{\partial \eta} + (\alpha - 1)\,\chi - 2\xi \frac{\partial \chi}{\partial \xi} - \sigma \frac{\partial \chi}{\partial \eta} = Lu + 2\,(2 - \alpha)\,\xi \frac{\partial u}{\partial \xi} - 2\alpha\sigma \frac{\partial u}{\partial \eta} - 2\frac{\partial v}{\partial \eta} + (\alpha^2 - 1)\, u, \tag{5}$$

$$\frac{\partial v}{\partial \xi} + \frac{1}{2}\frac{\sigma}{\xi}\frac{\partial v}{\partial \eta} + \frac{\partial \chi}{\partial \eta} = Lv + 2\,(2 - \alpha)\,\xi \frac{\partial v}{\partial \xi} - 2\alpha\sigma \frac{\partial v}{\partial \eta} + 2\frac{\partial u}{\partial \eta} + (\alpha^2 - 1)\, v,$$

$$\frac{\partial w}{\partial \xi} + \frac{1}{2}\frac{\sigma}{\xi}\frac{\partial w}{\partial \eta} = Lw + 4\,(2 - \alpha)\,\xi \frac{\partial w}{\partial \xi} - 4\,(\alpha - 1)\,\sigma \frac{\partial w}{\partial \eta} + (1 - \alpha)^2\, w,$$

where the operator $L = 4\xi^2 \frac{\partial^2}{\partial \xi^2} + 4\sigma\xi \frac{\partial^2}{\partial \xi \partial \eta} + (1 + \sigma^2)\frac{\partial^2}{\partial \eta^2}$. If $\xi \ge 0$, the time changes in the interval $0 < t < \infty$ (vortex genesis); however, if $\xi \le 0$, then $-\infty < t < 0$ (vortex collapse). The periodicity conditions (4) represent boundary-value conditions for solving the problem in the region of its definition: $0 \le \eta \le 2\pi$, $-\infty < z < \infty$.

The equation that determines the axial velocity component w is solved independently of the remaining ones. The variable z in this equation is "frozen," i.e., it is a parameter. In the columnar vortex approximation considered, when $r \to 0$, the dependence of the solution on z has not been revealed, whereas in the conical vortex it is appreciable. We will single out two flow schemes important for practice. If $w = az$, then on the plane $z = 0$ the nonflow condition is fulfilled, and the plane is the symmetry plane. If $w = az^2$, then $u \sim z$, $v \sim z$, and the sticking condition is fulfilled on the plane $z = 0$; the solution represents a primitive model of a tornado. Here we note that the flow in a real tornado, just as the flow considered, does not possess axial symmetry [6]. However, up to the present time erroneous publications have been based on the assumption of the axisymmetric nature of the flow [7].

System (5) has eigensolutions at $\partial w/\partial z \equiv 0$ and forced solutions induced by the axial velocity gradient $\partial w/\partial z \ne 0$.

We will represent the solution of system (5) in the form of a Fourier series:

$$u = u_k(\xi, z)\exp(ik\eta), \quad v = v_k(\xi, z)\exp(ik\eta), \quad w = w_k(\xi, z)\exp(ik\eta), \quad \chi = \chi_k(\xi, z)\exp(ik\eta).$$

Here summation over the parameter k that takes an integer value is meant. With such a representation the periodicity condition (4) is fulfilled automatically.

The solutions for w_k have the form $w_k = c(z)w_{0k}(\xi)$, where c is an arbitrary complex function of the real variable z. The dependence on z will disappear if the variables u_k, v_k, and χ_k are extended dc/dz times. Then, assuming that $u_k = u_k(\xi)$, $v_k = v_k(\xi)$, $w_k = w_k(\xi)$, and $\chi_k = \chi_k(\xi)$, we reduce the problem to a system of ordinary differential equations:

$$ikv_k + w_{0k} - 2\xi u_k' + (1 + \alpha - ik\sigma)\, u_k = 0,$$

$$(\alpha - 1 - ik\sigma)\,\chi_k - 2\xi\chi_k' = Lu_k + (4\xi - 2\alpha\xi - 1)\,u' + \left(\alpha^2 - 1 - 2ik\alpha\sigma - \frac{1}{2}\frac{ik\sigma}{\xi}\right)u_k - 2ikv_k\,,$$

$$ik\chi_k = Lv_k + (4\xi - 2\alpha\xi - 1)\,v_k' + \left(\alpha^2 - 1 - 2ik\alpha\sigma - \frac{ik\sigma}{2\xi}\right)v_k + 2iku_k\,, \tag{6}$$

$$Lw_{0k} + (8\xi - 4\alpha\xi - 1)\,w_{0k}' + \left[(\alpha - 1)^2 - 4ik\sigma\,(\alpha - 1) - \frac{ik\sigma}{2\xi}\right]w_{0k} = 0\,,$$

where the operator $Ly = 4\xi^2 y'' + 4ik\sigma\xi y' - k^2(1 + \sigma^2)y$; the derivatives with respect to ξ are primed. The initial data to the Cauchy problem are determined from the expansion of the solution for $\xi \to 0$. The expansion for w_{0k} has the form $w_{0k} = O[\xi^2 \exp(-1/4\xi)]$. According to the general theorems of the theory of differential equations [8], a solution of the linear system (6) exists in the entire range of change of ξ from 0 to ∞.

Stationary focus. In a stationary case $0 \le \eta = \theta - \sigma \ln r \le 2\pi$ the desired functions do not depend on ξ. The system of stationary equations results from Eq. (5), if on the left-hand sides of the equations of motion we formally assume that $\partial/\xi = 0$ and $\xi = \infty$. The system of ordinary differential equations for the amplitudes of the Fourier series will be transformed into a system of algebraic equations:

$$(1 + \alpha - ik\sigma)\,u_k + ikv_k + w_{0k} = 0\,, \quad (\alpha - 1 - ik\sigma)\,\chi_k = [(\alpha^2 - 1) - k^2(1 + \sigma^2) - 2ik\alpha\sigma]\,u_k - 2ikv_k\,,$$

$$ik\chi_k = [(\alpha^2 - 1) - k^2(1 + \sigma^2) - 2ik\alpha\sigma]\,v_k + 2iku_k\,, \quad [k^2(1 + \sigma^2) - (1 - \alpha)^2 + 4ik\sigma\,(\alpha - 1)]\,w_{0k} = 0\,. \tag{7}$$

The periodic solution of the last equation in (7) exists only in the degenerate case of $\sigma = 0$ that corresponds not to a spiral, but rather to an axisymmetric flow. At $\alpha = 1$ there exists a one-dimensional flow $w = w(z)$; therefore in the principal terms the forced solution is not periodic.

It remains to consider the case of characteristic solutions of the homogeneous system (7). Its characteristic equation does not have real solutions for integers $k \ne 0$. Consequently, in a stationary case the solution of system (6) at $\sigma \ne 0$ does not exist.

Limiting cycle. We will consider a nonstationary solution in the vicinity of the stationary limiting cycle $r_* = 1$. The expansion of the desired functions is similar to (2):

$$u_r(r, \theta, z, t) = x^{\alpha} u(\xi, \eta, z) + o(x^{\alpha})\,, \quad u_\theta(r, \theta, z, t) = x^{\alpha-1} v(\xi, \eta, z) + o(x^{\alpha-1})\,,$$

$$u_z(r, \theta, z, t) = x^{\alpha-1} w(\xi, \eta, z) + o(x^{\alpha-1})\,, \quad p(r, \theta, z, t) = x^{\alpha-1} P(\xi, \eta, z) + o(x^{\alpha-1})\,, \tag{8}$$

where $x = r - 1$; $\eta = \theta - \sigma \ln x$, and $\xi = tx^{-2}$.

On the right-hand side of Eq. (1) the second derivatives with respect to r dominate. The velocity components u_θ and u_r are determined from the heat conduction equation independently of the remaining functions. Thus, to determine w from (1) we obtain the equation

$$4\xi^2 \frac{\partial^2 w}{\partial \xi^2} + 4\sigma\xi \frac{\partial^2 w}{\partial \xi \partial \eta} + \sigma^2 \frac{\partial^2 w}{\partial \eta^2} + [2\xi(5 - 2\alpha) - 1]\frac{\partial w}{\partial \xi} + (3 - 2\alpha)\,\sigma \frac{\partial w}{\partial \eta} + (\alpha - 1)(\alpha - 2)\,w = 0\,. \tag{9}$$

The quantity $v(\xi, \eta, z)$ is determined from the same equation. Since in Eq. (9) the dependence on z is not evident, it reveals itself parametrically — the constants entering into this equation are actually the functions of z. For a solitary harmonic we have

$$w = c(z)\,w_k(\xi)\exp(ik\eta)\,,$$

where the function $c(z)$ is arbitrary and $w_{k0}(\xi)$ is determined from the ordinary differential equation:

$$4\xi^2 w_k'' + [2\xi(5-2\alpha) - 1 + 4ik\sigma\xi]\, w_k' + b w_k = 0\,, \tag{10}$$

where $b = (\alpha-1)(\alpha-2) - k^2\sigma^2 + ik\sigma(3-2\alpha)$. This equation does not have a stationary solution, since $b \neq 0$.

The initial data ($\xi = 0$) for the nonstationary problem can be of two types: 1) increasing linearly over ξ from a certain constant value, and 2) zero ones proportional to $\exp\left(-\frac{1}{4\xi}\right)$.

Thus, the heat conduction equation has spiral-periodic solutions, and the continuity equation

$$\frac{\partial v}{\partial \eta} - 2\xi\frac{\partial v}{\partial \xi} + \frac{\partial w}{\partial z} - \sigma\frac{\partial u}{\partial \eta} + \alpha u = 0$$

has the periodic solution $u = c' u_k(\xi)\exp(ik\eta)$, $v = c' v_k(\xi)\exp(ik\eta)$. The function $v_k(\xi)$, just as $w_k(\xi)$, satisfies Eq. (10) and $u_k(\alpha - ik\sigma) = 2\xi v_k' - ikv_k - w_k$.

The pressure gradient is equal to

$$\frac{\partial p(r, \theta, z, t)}{\partial r} = \frac{\partial^2 u_r}{\partial z^2} - \frac{\partial u_r}{\partial t}\,.$$

Conical Vortex. When $R \to 0$ (see Fig. 1b), the singularity comes into play different from that typical of a cylindrical vortex. To study this singularity, it is first of all necessary to select the trajectory of the approach to singularity. Let $R \to 0$, $\vartheta \to 0$, and $\varphi \to \infty$. One has to transform from three spherical coordinates R, ϑ, and φ to two auxiliary coordinates μ and β, where β characterizes the relative velocity of the striving of ϑ and R to zero. In the most general case, these velocities are identical in order of magnitude; therefore we assume that $\beta = \vartheta/R$. Since with $R \to 0$ the viscous terms (the right-hand side of Eq. (1)) has the order of $O(R^{-4})$, the influence of the nonstationarity is substantial when $t = O(R^4)$. Therefore we introduce the self-similar coordinate $\tau = tR^{-4}$; then the coordinate expansion will take the form

$$u_\vartheta = R^{\alpha+1} v(\beta, \mu, \tau) + o(R^{\alpha+1})\,, \quad u_\varphi = R^{\alpha+1} w(\beta, \mu, \tau) + o(R^{\alpha+1})\,,$$
$$u_R = R^{\alpha} u(\beta, \mu, \tau) + o(R^{\alpha})\,, \quad p = R^{\alpha-1}\chi(\beta, \mu, \tau) + o(R^{\alpha-1})\,. \tag{11}$$

We will consider again the logarithmic spiral $\varphi_0 = \sigma \ln \vartheta$. From Eq. (1) we obtain

$$4\beta\tau\frac{\partial u}{\partial \tau} + \beta^2\frac{\partial u}{\partial \beta} - \beta\frac{\partial v}{\partial \beta} + \sigma\frac{\partial v}{\partial \mu} - \frac{\partial w}{\partial \mu} - \beta(2+\alpha)u - v = 0\,,$$
$$\beta^2\frac{\partial v}{\partial \tau} + \beta^2\frac{\partial \chi}{\partial \beta} - \sigma\beta\frac{\partial \chi}{\partial \mu} = Mv - 2\frac{\partial w}{\partial \mu} + 2\beta^2\frac{\partial u}{\partial \beta} - 2\sigma\beta\frac{\partial u}{\partial \mu}\,, \tag{12}$$
$$\beta^2\frac{\partial w}{\partial \tau} + \beta\frac{\partial \chi}{\partial \mu} = Mw + 2\frac{\partial v}{\partial \mu} + 2\beta\frac{\partial u}{\partial \mu}\,, \quad \beta^2\frac{\partial u}{\partial \tau} - (M+1)u = 0\,,$$

where the operator

$$M = (1+\sigma^2)\frac{\partial^2}{\partial \mu^2} - 2\sigma\beta\frac{\partial^2}{\partial \beta\partial \mu} + \beta^2\frac{\partial^2}{\partial \beta^2} + \beta\frac{\partial}{\partial \beta} - 1\,.$$

The longitudinal pressure gradient is absent, and the equation for the longitudinal velocity component u is solved independently of the remaining ones.

As before, the periodicity conditions are satisfied by the solution in the form of the Fourier series:

$$u(\beta, \mu, \tau) = \sum_{k=0}^{\infty} u_k(\beta, \tau) \exp(ik\mu) , \quad v(\beta, \mu, \tau) = \sum_{k=0}^{\infty} v_k(\beta, \tau) \exp(ik\mu) ,$$

$$w(\beta, \mu, \tau) = \sum_{k=0}^{\infty} w_k(\beta, \tau) \exp(ik\mu) , \quad \chi(\beta, \mu, \tau) = \sum_{k=0}^{\infty} \chi_k(\beta, \tau) \exp(ik\mu) .$$

From Eq. (12) we find

$$4\beta\tau \frac{\partial u_k}{\partial \tau} + \beta^2 \frac{\partial u_k}{\partial \beta} - \beta \frac{\partial v_k}{\partial \beta} - ikw_k - \beta(2+\alpha) u_k - v_k (1 - ik\sigma) = 0 ,$$

$$\beta^2 \frac{\partial v_k}{\partial \tau} + \beta^2 \frac{\partial \chi_k}{\partial \beta} - ik\beta\sigma\chi_k = Mv_k - 2ikw_k + 2\beta^2 \frac{\partial u_k}{\partial \beta} - 2ik\beta\sigma u_k ,$$

$$\beta^2 \frac{\partial w_k}{\partial \tau} + ik\beta\chi_k = Mw_k + 2ikv_k + 2ik\beta u_k , \quad \beta^2 \frac{\partial u_k}{\partial \tau} - Mu_k - u_k = 0 , \tag{13}$$

where

$$M = \beta^2 \frac{\partial^2}{\partial \beta^2} + (1 - 2ik\sigma) \beta \frac{\partial}{\partial \beta} - 1 - k^2 (1 + \sigma^2) .$$

The last equation of this system is independent of the remaining ones and has a solution of the type of a running wave: $u_k = a_0(\beta) \exp(i\omega\tau)$, where $a_0(\beta)$ is determined from the Bessel equation in complex variables with a limited solution on the axis $\beta = 0$. The remaining velocity components v_k and w_k also correspond to the running wave.

Stationary focus. In a stationary case ($\partial/\partial\tau = 0$) system (13) is transformed into a system of ordinary differential equations. Since it is homogeneous in the variable β, its accurate solution can be represented in an exponential form:

$$u_k = a_1\beta^{n_1} + a_2\beta^{n_2} , \quad v_k = b_1\beta^{n_1+1} + b_2\beta^{n_2+1} , \quad w_k = c_1\beta^{n_1+1} + c_2\beta^{n_2+1} ,$$

$$\chi_k = \chi_1\beta^{n_1} + \chi_2\beta^{n_2} , \quad n_{1,2}/k = i\sigma \pm 1 .$$

The characteristic ($u_k \equiv 0$) stationary solution can also be presented in an exponential form.

Limiting cycle. We will consider a flow in the vicinity of the conical limiting cycle $\vartheta = \vartheta_0 = \text{const}$ ($\vartheta \to \vartheta_0$, $R \to 0$, $\varphi \to 0$). We introduce the coordinates: $\beta = (\vartheta \to \vartheta_0)/R$, $0 \le \mu = \varphi - \sigma \ln |\vartheta \to \vartheta_0| \le 2\pi$, $\tau = tR^{-4}$.

The expansions for all the functions, except for the circumferential velocity component u_φ, are identical to (11):

$$u_\vartheta = R^{\alpha+1} v(\beta, \mu, \tau) + o(R^{\alpha+1}) , \quad u_\varphi = R^{\alpha} w(\beta, \mu, \tau) + o(R^{\alpha}) ,$$

$$u_R = R^{\alpha} u(\beta, \mu, \tau) + o(R^{\alpha}) , \quad p = R^{\alpha-1} \chi(\beta, \mu, \tau) + o(R^{\alpha-1}) .$$

The continuity equation takes the form

$$\frac{\partial v}{\partial \beta} - \frac{\sigma}{\beta} \frac{\partial v}{\partial \mu} + \frac{1}{\sin \vartheta_0} \frac{\partial w}{\partial \mu} + \sigma \frac{\partial u}{\partial \mu} - \beta \frac{\partial u}{\partial \beta} - 4\tau \frac{\partial u}{\partial \tau} + (2+\alpha) u = 0 .$$

The velocity components u and w are determined independently of each other from the heat conduction equation. Thus, the velocity u is determined from the equation

$$\beta^2 \frac{\partial u}{\partial \tau} = \beta^2 \frac{\partial^2 u}{\partial \beta^2} - 2\beta\sigma \frac{\partial^2 u}{\partial \beta \partial \mu} + \sigma^2 \frac{\partial^2 u}{\partial \mu^2} + \sigma \frac{\partial u}{\partial \mu} .$$

The solution exists in the form of the running wave:

$$u(\beta, \mu, \tau) = u_1(\beta) \exp i(\omega\tau + k\mu) , \quad w(\beta, \mu, \tau) = w_1(\beta) \exp i(\omega\tau + k\mu) ,$$

$$v(\beta, \mu, \tau) = [\tau v_1(\beta) + v_2(\beta)] \exp i(\omega\tau + k\mu) .$$

The pressure amplitude and the transversal velocity v increase with time. In the case of a quasi-two-dimensional ($u \equiv 0$) flow the amplitudes of the running wave are constant.

Conclusions. A geometric classification of spiral vortices is given. Three-dimensional spiral vortices are subdivided into columnar and conical ones, whereas the two-dimensional vortices — into sloping, logarithmic, and steep ones. The spiral vortex has a singularity of the type of either a focus or of the limiting cycle. In an inviscid flow the spiral vortex includes spiral discontinuities of the tangential velocity component [4]. In a viscous flow there are no discontinuities, and the functions depend continuously on the spiral coordinate. The possibility of the existence of spiral vortices in a creeping flow proved on specific examples within the framework of the local theory means that their structure is not decisively determined by the nonlinearity. Spiral coherent vortices that play an important role in a developed turbulent flow [4, 9] exist in the entire range of scales, i.e., of Re numbers.

NOTATION

p, pressure; Re, Reynolds number; s, spiral length; t, time; **u**, velocity vector; α, exponent; β, dimensionless length of spiral arc; η, μ, spiral coordinates; σ, coefficient of spiral steepness; τ, ξ, self-similar coordinates; r, θ, z, axes of cylindrical coordinate system; R, ϑ, φ, axes of spherical coordinate system; ∇, gradient operator; Δ, Laplace operator.

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